A remark on left invariant metrics on compact Lie groups

Lorenz J. Schwachhöfer*

May 7, 2007

1 Introduction

The investigation of manifolds with non-negative sectional curvature is one of the classical fields of study in global Riemannian geometry. While there are few known obstruction for a closed manifold to admit metrics of non-negative sectional curvature, there are relatively few known examples and general construction methods of such manifolds (see [Z] for a detailed survey).

In this context, it is particularly interesting to investigate left invariant metrics on a compact connected Lie group G with Lie algebra $\mathfrak{g}$. These metrics are obtained by left translation of an inner product on $\mathfrak{g}$. If this metric is biinvariant then its sectional curvature is non-negative, and it is known that the set of inner products on $\mathfrak{g}$ whose corresponding left invariant metric on G has non-negative sectional curvature is a connected cone; indeed, each such inner product can be connected to a biinvariant one by a canonical path ([T]).

In the present article, it is shown that the stretching of the biinvariant metric in the direction of a subalgebra of $\mathfrak{g}$ almost always produces some negative sectional curvature of the corresponding left invariant metric on G. In fact, the following theorem answers a question raised in [Z, Problem 1, p.9].

Theorem 1.1 *Let $H \subset G$ be compact Lie groups with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, let Q be a biinvariant inner product on $\mathfrak{g}$, and for $t > 0$ let g_t be the left invariant metric on G induced by the inner product*

$$Q_t := t\ Q|_{\mathfrak{h}} + Q|_{\mathfrak{h}^\perp}. \tag{1}$$

If there is a $t > 1$ such that g_t has non-negative sectional curvature, then then the semi-simple part of $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Note that this condition is indeed optimal: if $t \leq 1$ then g_t is known to have non-negative sectional curvature, and if the semi-simple part of $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then g_t has non-negative sectional curvature even for $t \leq 4/3$ ([GZ]).

There is yet another reason why this result is of interest. One of the most spectacular source of examples of manifolds of non-negative sectional curvature of the last decade was given in [GZ] where it was shown that any closed cohomogeneity one manifold whose non-principal orbits have codimension at most two admit invariant metrics of non-negative sectional curvature. Their construction is

*Research supported by the Schwerpunktprogramm Differentialgeometrie of the Deutsche Forschungsgesellschaft

based on glueing homogeneous disk bundles of rank ≤ 2 along a totally geodesic boundary which is equipped with a normal homogeneous metric.

The reason for this construction to work is due to the fact that the structure group of the fibers is contained in $H = SO(k)$ where k is the rank of the bundle. If $k \leq 2$, then H is abelian, so that the metrics g_t from Theorem 1.1 have non-negative sectional curvature for some $t > 1$.

Our result now suggests that for most subgroups $H' \subset H$, the metric on G/H' induced by the metric g_t with $t > 1$ from Theorem 1.1 will have some negative sectional curvature as well. Therefore, it will be difficult to find more examples of non-negatively curved metrics on homogeneous vector bundles over G/H with normal homogeneous collar. Also, note that there are examples of cohomogeneity one manifolds, including the Kervaire spheres, which do not admit invariant metric of non-negative sectional curvature at all ([GVWZ]).

2 Proof of Theorem 1.1

Let $H \subset G$, $\mathfrak{h} \subset \mathfrak{g}$, Q_t and g_t be as in Theorem 1.1, and let $\mathfrak{m} := \mathfrak{h}^\perp$, so that we have the orthogonal splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{2}$$

Then a calculation shows that for any $s > 0$ and $t := s/(1+s)$, the multiplication map

$$(H \times G, sQ|_{\mathfrak{h}} + Q|_{\mathfrak{g}}) \longrightarrow (G, g_t) \tag{3}$$

becomes a Riemannian submersion (cf. e.g. [Ch]). But $sQ|_{\mathfrak{h}} + Q|_{\mathfrak{g}}$ is a biinvariant metric on $H \times G$ which therefore has non-negative sectional curvature, and by O'Neill's formula so does Q_t. Since $Q_1 = Q$ is a biinvariant metric, and any $t \in (0,1)$ can be written as $t = s/(1+s)$ for some $s > 0$, we conclude that Q_t has non-negative sectional curvature for all $t \leq 1$.

We shall divide the proof of Theorem 1.1 into two lemmas.

Lemma 2.1 *Suppose that the metric Q_t on G has non-negative sectional curvature for some $t > 1$. Then for all $x, y \in \mathfrak{g}$ with $[x,y] = 0$ we must have $[x_{\mathfrak{h}}, y_{\mathfrak{h}}] = 0$, where $x = x_{\mathfrak{h}} + x_{\mathfrak{m}}$ and $y = y_{\mathfrak{h}} + y_{\mathfrak{m}}$ is the decomposition according to (2).*

Proof. The curvature tensor R^t of the metric g_t has been calculated e.g. in [GZ]. Namely, for elements $x = x_{\mathfrak{h}} + x_{\mathfrak{m}}$ and $y = y_{\mathfrak{h}} + y_{\mathfrak{m}}$ we have

$$\begin{aligned} Q_t(R^t(x,y)y,x) \;\; = \;\; & \tfrac{1}{4}||\ [x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{m}} + t[x_{\mathfrak{h}}, y_{\mathfrak{m}}] + t[x_{\mathfrak{m}}, y_{\mathfrak{h}}]\ ||_Q^2 \\ + \;\; & \tfrac{1}{4}t||\ [x_{\mathfrak{h}}, y_{\mathfrak{h}}]\ ||_Q^2 + \tfrac{1}{2}t(3-2t)Q([x_{\mathfrak{h}}, y_{\mathfrak{h}}], [x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{h}}) + (1 - \tfrac{3}{4}t)||\ [x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{h}}\ ||_Q^2. \end{aligned} \tag{4}$$

Let $x^t := tx_{\mathfrak{h}} + x_{\mathfrak{m}}$ and $y^t := ty_{\mathfrak{h}} + y_m$. Then, using that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, it follows that

$$[x^t, y^t]_{\mathfrak{h}} = t^2[x_{\mathfrak{h}}, y_{\mathfrak{h}}] + [x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{h}} \qquad \text{and} \qquad [x^t, y^t]_{\mathfrak{m}} = [x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{m}} + t[x_{\mathfrak{h}}, y_{\mathfrak{m}}] + t[x_{\mathfrak{m}}, y_{\mathfrak{h}}].$$

If we assume that $[x^t, y^t] = 0$, then $[x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{h}} = -t^2[x_{\mathfrak{h}}, y_{\mathfrak{h}}]$ and $[x_{\mathfrak{m}}, y_{\mathfrak{m}}]_{\mathfrak{m}} + t[x_{\mathfrak{h}}, y_{\mathfrak{m}}] + t[x_{\mathfrak{m}}, y_{\mathfrak{h}}] = 0$. Substituting this into (4) yields

$$\begin{aligned} Q_t(R^t(x,y)y,x) &= \left(\tfrac{1}{4}t - \tfrac{1}{2}t^3(3-2t) + (1-\tfrac{3}{4}t)t^4\right) ||\ [x_{\mathfrak{h}}, y_{\mathfrak{h}}]\ ||_Q^2 \\ &= -\tfrac{1}{4}t(t-1)^3(1+3t)||\ [x_{\mathfrak{h}}, y_{\mathfrak{h}}]\ ||_Q^2. \end{aligned} \tag{5}$$

If this expression is non-negative for some $t > 1$, then $[x_{\mathfrak{h}}, y_{\mathfrak{h}}] = 0$. Thus, $[x^t_{\mathfrak{h}}, y^t_{\mathfrak{h}}] = t^2[x_{\mathfrak{h}}, y_{\mathfrak{h}}] = 0$ whenever $[x^t, y^t] = 0$. ∎

Lemma 2.2 *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra such that all $x, y \in \mathfrak{g}$ with $[x, y] = 0$ satisfy $[x_{\mathfrak{h}}, y_{\mathfrak{h}}] = 0$, where $x = x_{\mathfrak{h}} + x_{\mathfrak{m}}$ and $y = y_{\mathfrak{h}} + y_{\mathfrak{m}}$ is the decomposition according to (2). Then the semi-simple part of $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r$ be the decomposition into the center and simple ideals. Then $[x_{\mathfrak{h}}, y_{\mathfrak{h}}] = 0$ iff $[x_{\mathfrak{h}_k}, y_{\mathfrak{h}_k}] = 0$ for all k. Also, the semi-simple part of $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ iff $\mathfrak{h}_k \lhd \mathfrak{g}$ for all k. Thus, it sufices to show the lemma for all $\mathfrak{h}_k$, whence we shall assume for the rest of the proof that $\mathfrak{h}$ is *simple*.

Step 1. Let $y \in \mathfrak{m}$ be such that there is an $0 \neq x \in \mathfrak{h}$ with $[x, y] = 0$. Then $[\mathfrak{h}, y] = 0$.

For any $a \in \mathfrak{m}$ and $t \in \mathbb{R}$, we have $[Ad_{\exp(ta)}x, Ad_{\exp(ta)}y] = Ad_{\exp(ta)}[x, y] = 0$, hence by hypothesis $[(Ad_{\exp(ta)}x)_{\mathfrak{h}}, (Ad_{\exp(ta)}y)_{\mathfrak{h}}] = 0$.

But $[a, x] \in [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$, hence $(Ad_{\exp(ta)}x)_{\mathfrak{h}} = x + O(t^2)$, whereas $(Ad_{\exp(ta)}y)_{\mathfrak{h}} = t[a, y]_{\mathfrak{h}} + \frac{1}{2}t^2[a, [a, y]]_{\mathfrak{h}} + O(t^3)$. Therefore, for all $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 &= \left[(Ad_{\exp(ta)}x)_{\mathfrak{h}}, (Ad_{\exp(ta)}y)_{\mathfrak{h}}\right] = t[x, [a, y]_{\mathfrak{h}}] + \tfrac{1}{2}t^2[x, [a, [a, y]]_{\mathfrak{h}}] + O(t^3) \\ &= t[x, [a, y]]_{\mathfrak{h}} + \tfrac{1}{2}t^2[x, [a, [a, y]]]_{\mathfrak{h}} + O(t^3). \end{aligned} \tag{6}$$

The last equation follows since for all $x \in \mathfrak{h}$ and $z = z_{\mathfrak{h}} + z_{\mathfrak{m}}$ we have $[x, z_{\mathfrak{h}}] \in \mathfrak{h}$ and $[x, z_{\mathfrak{m}}] \in \mathfrak{m}$, whence $[x, z_{\mathfrak{h}}] = [x, z]_{\mathfrak{h}}$. Thus, we must have $[x, [a, y]]_{\mathfrak{h}} = 0$ for all $a \in \mathfrak{m}$. On the other hand, if $a \in \mathfrak{h}$ then $[x, [a, y]] \in [\mathfrak{h}, [\mathfrak{h}, \mathfrak{m}]] \subset \mathfrak{m}$, hence $[x, [a, y]]_{\mathfrak{h}} = 0$ for all $a \in \mathfrak{h}$ as well, and therefore,

$$0 = Q([x, [\mathfrak{g}, y]], \mathfrak{h}) = Q(\mathfrak{g}, [[x, \mathfrak{h}], y]), \quad \text{i.e.,} \quad [[x, \mathfrak{h}], y] = 0. \tag{7}$$

By [S, Lemma 4.4] and the simplicity of $\mathfrak{h}$, it follows that $\mathfrak{h}$ is the linear span of x, $[x, \mathfrak{h}]$ and $[[x, \mathfrak{h}], [x, \mathfrak{h}]]$. Since $[x, y] = 0$, and $[[x, \mathfrak{h}], y] = 0$ by (7), this together with the Jacobi identity now implies that $[\mathfrak{h}, y] = 0$ as claimed.

Step 2. Let $y \in \mathfrak{m}$ be such that $[\mathfrak{h}, y] = 0$. Let $\mathfrak{g}' \lhd \mathfrak{g}$ and $\mathfrak{g}'' \lhd \mathfrak{g}$ be the ideals generated by $\mathfrak{h}$ and y, respectively. Then $Q(\mathfrak{g}', \mathfrak{g}'') = 0$ and $[\mathfrak{g}', \mathfrak{g}''] = 0$. In particular, $Q(\mathfrak{g}', y) = 0$

First, note the it suffices to show that $Q(\mathfrak{h}, \mathfrak{g}'') = 0$. For if this is the case, it then follows that $Q(ad(\mathfrak{g})^n(\mathfrak{h}), \mathfrak{g}'') = Q(\mathfrak{h}, ad(\mathfrak{g})^n(\mathfrak{g}'')) = Q(\mathfrak{h}, \mathfrak{g}'') = 0$, which implies that $Q(\mathfrak{g}', \mathfrak{g}'') = 0$. Hence, $Q([\mathfrak{g}', \mathfrak{g}''], \mathfrak{g}) = Q(\mathfrak{g}', [\mathfrak{g}'', \mathfrak{g}]) = Q(\mathfrak{g}', \mathfrak{g}'') = 0$ so that $[\mathfrak{g}', \mathfrak{g}''] = 0$ follows.

By [S, Lemma 4.4], $\mathfrak{g}''$ is the linear span of y, $[\mathfrak{g}, y]$ and $[\mathfrak{g}, [\mathfrak{g}, y]]$. Since $y \in \mathfrak{m}$, we have $Q(y, \mathfrak{h}) = 0$, and $Q([\mathfrak{g}, y], \mathfrak{h}) = Q(\mathfrak{g}, [\mathfrak{h}, y]) = 0$ by hypothesis. Thus, $Q(\mathfrak{h}, \mathfrak{g}'') = 0$ will be demonstrated once we show that $Q([\mathfrak{g}, [\mathfrak{g}, y]], \mathfrak{h}) = 0$.

For a fixed $h \in \mathfrak{h}$, we define the bilinear form α_h on $\mathfrak{g}$ by

$$\alpha_h(a, b) := Q([a, [b, y]], h).$$

Thus, our goal shall be to show that $\alpha_h = 0$ for all $h \in \mathfrak{h}$. Note that $\alpha_h(a,b) - \alpha_h(b,a) = Q([a,[b,y]] - [b,[a,y]], h) = Q([[a,b],y],h) = -Q([a,b],[h,y]) = 0$ by hypothesis, hence α_h is *symmetric*. If $b \in \mathfrak{h}$, then $[b,y] = 0$ by hypothesis, so that $\alpha_h(\mathfrak{g}, \mathfrak{h}) = 0$.

By or hypothesis and step 1, (6) holds for *all* $x \in \mathfrak{h}$, thus the vanishing of the t^2-coefficient of (6) implies that

$$0 = Q([\mathfrak{h}, [a,[a,y]], \mathfrak{h}) = Q([a,[a,y]], [\mathfrak{h}, \mathfrak{h}]) = Q([a,[a,y]], \mathfrak{h}) \text{ for all } a \in \mathfrak{m}.$$

Thus, $\alpha_h(a,a) = 0$ for all $a \in \mathfrak{m}$ and therefore, $\alpha_h = 0$ for all $h \in \mathfrak{h}$ as asserted.

Step 3. $\mathfrak{h} \lhd \mathfrak{g}$.

Let $\mathfrak{g}' \lhd \mathfrak{g}$ be the ideal generated by $\mathfrak{h}$. By steps 1 and two, it follows that there cannot be an $0 \neq x \in \mathfrak{h}$ and $0 \neq y \in \mathfrak{m} \cap \mathfrak{g}'$ with $[x,y] = 0$. This immediately implies that $rk(\mathfrak{h}) = rk(\mathfrak{g}')$.

If $rk(\mathfrak{h}) = rk(\mathfrak{g}') = 1$, then $\mathfrak{h} = \mathfrak{g}' \lhd \mathfrak{g}$ and we are done. If $rk(\mathfrak{h}) \geq 2$ then we can choose linearly independent elements $x_1, x_2 \in \mathfrak{h}$ with $[x_1, x_2] = 0$. If $\mathfrak{m} \cap \mathfrak{g}' \neq 0$, then the restrictions of ad_{x_i} to $\mathfrak{m} \cap \mathfrak{g}'$ have common eigenspaces, i.e., there is an orthogonal decomposition

$$\mathfrak{m} \cap \mathfrak{g}' = V_1 \oplus \ldots \oplus V_m$$

into two-dimensional subspaces V_k on which both ad_{x_i} act by a multiple of rotation by a right angle. Therefore, for each k, there is a suitable $0 \neq x^k \in span(x_1, x_2) \subset \mathfrak{h}$ such that $[x^k, V_k] = 0$ which is a contradiction. Therefore, $\mathfrak{m} \cap \mathfrak{g}' = 0$, i.e., $\mathfrak{h} = \mathfrak{g}' \lhd \mathfrak{g}$. ∎

References

[Ch] J. Cheeger, *Some examples of manifolds of non-negative curvature*, J. Diff. Geom. 8 (1973) 623–628

[GVWZ] K.Grove, L.Verdiani, B.Wilking and W.Ziller, *Non-negative curvature obstruction in cohomogeneity one and the Kervaire spheres*, Ann. del. Scuola Norm. Sup. **5**159-170 (2006)

[GZ] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) 152 (2000) 331–36

[S] L.J. Schwachhöfer, *On the Ricci curvature of normal metrics on biquotients*, (preprint) available at arXiv:math.DG/0702612 (2007)

[ST] L.J. Schwachhöfer, W. Tuschmann, *Metrics of positive Ricci curvature on quotient spaces*, Math. Ann. **330** (1), 59 – 91 (2004)

[T] K. Tapp, *Homogeneous metrics with nonnegative curvature*, (preprint)

[Z] W. Ziller, *Examples of Riemannian mannifolds with non-negative sectional curvature*, (preprint) available at arXiv:math.DG/0701389 (2007)

Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany
Email: lschwach@math.uni-dortmund.de